ON THE CLASSIFICATION OF PRINCIPAL PU_2-BUNDLES OVER A 6-COMPLEX

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ABSTRACT. We point out and correct an error in L. M. Woodward's 1982 paper "The classification of principal PU_n-bundles over a 4-complex."

In recent investigations [1] into topological Azumaya algebras, we have been very fortunate to have the insights provided by Woodward's paper [5]. However, it turns out that one part of the main theorem is slightly incorrect, because of the mistaken assumption that $\pi_5 \mathrm{B}\,\mathrm{PU}_2 = 0$, which appears on p.521. Here, $\mathrm{PU}_2 \simeq \mathrm{SU}_2/\mu_2$, the quotient of SU_2 by its center μ_2, and $\mathrm{B}\,\mathrm{PU}_2$ is the classifying space of PU_2. Thus, if X is a CW complex, the set $[X, \mathrm{B}\,\mathrm{PU}_2]$ of homotopy classes of maps $X \to \mathrm{B}\,\mathrm{PU}_2$ classifies principal PU_2-bundles on X. In fact, as shown by Bott [3, Theorem 5], $\pi_5 \mathrm{B}\,\mathrm{PU}_2 = \mathbb{Z}/2$. We explain how this affects the main theorem of [5], and how to correct the theorem.

Given a PU_n-bundle $\xi : X \to \mathrm{B}\,\mathrm{PU}_n$, [5] constructs two characteristic classes: a class $t(\xi) \in \mathrm{H}^2(X, \mathbb{Z}/2)$ and a class $q(\xi) \in \mathrm{H}^4(X, \mathbb{Z})$. Write ρ_{2n*} for the operation $\mathrm{H}^4(\cdot, \mathbb{Z}) \to \mathrm{H}^4(\cdot, \mathbb{Z}/2n)$ induced by the quotient map $\mathbb{Z} \to \mathbb{Z}/2n$, and let $C : \mathrm{H}^2(\cdot, \mathbb{Z}/n) \to \mathrm{H}^4(\cdot, \mathbb{Z}/2n)$ be the Pontrjagin square. The characteristic classes $t(\xi)$ and $q(\xi)$ are related by the requirement that

$$
\text{(1)} \qquad \rho_{2n*}(q(\xi)) = \begin{cases} (n+1)C(t(\xi)) & \text{if } n \text{ is even,} \\ \frac{n+1}{2}C(t(\xi)) & \text{if } n \text{ is odd.} \end{cases}
$$

The main theorem of [5] consists of three parts. The first describes the image of

$$
[X, \mathrm{B}\,\mathrm{PU}_n] \to \mathrm{H}^2(X, \mathbb{Z}/n) \times \mathrm{H}^4(X, \mathbb{Z})
$$

when $\dim X \leq 6$: it is precisely the pairs of classes satisfying (1). The second part says that when $\dim X \leq 4$, for each $x \in \mathrm{H}^2(X, \mathbb{Z}/n)$, there is a PU_n-bundle ξ with $t(\xi) = x$. The third part says that when $\dim X \leq 4$, the map of sets $[X, \mathrm{B}\,\mathrm{PU}_n] \to \mathrm{H}^2(X, \mathbb{Z}/n) \times \mathrm{H}^4(X, \mathbb{Z})$ is injective when $\mathrm{H}^4(X, \mathbb{Z})$ has no p-torsion for p dividing $2n$.

Only a small portion of the theorem is false: when $\dim X = 6$ and $n = 2$, there are some classes $(x, y) \in \mathrm{H}^2(X, \mathbb{Z}/2) \times \mathrm{H}^4(X, \mathbb{Z})$ satisfying $\rho_{4*}(y) = 3C(x)$, but which are not the characteristic classes of any PU_2-bundle over X.

For 6-complexes X, there is a surjection $[X, \mathrm{B}\,\mathrm{PU}_n] \to [X, \mathrm{B}\,\mathrm{PU}_n[5]]$, where $\mathrm{B}\,\mathrm{PU}_n[5]$ denotes the 5th stage in the Postnikov tower for $\mathrm{B}\,\mathrm{PU}_n$. If $n > 2$, then $\mathrm{B}\,\mathrm{PU}_n[5] \simeq \mathrm{B}\,\mathrm{PU}_n[4]$, since $\pi_5 \mathrm{B}\,\mathrm{PU}_n = 0$ in this case. However, for $n = 2$, this is not the case, and we are left with the problem of computing the image of $[X, \mathrm{B}\,\mathrm{PU}_2[5]] \to [X, \mathrm{B}\,\mathrm{PU}_2[4]]$. The characteristic classes above are obtained by showing that $\mathrm{B}\,\mathrm{PU}_2[4]$ is equivalent to the homotopy fiber of the map $-3C + \rho_{4*}$

$$
\text{(2)} \qquad K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4) \to K(\mathbb{Z}/4, 4).
$$

Thus, given a PU_n-bundle ξ over X, the characteristic classes are given by the composition

$$
X \xrightarrow{\xi} \mathrm{B}\,\mathrm{PU}_2 \to \mathrm{B}\,\mathrm{PU}_2[4] \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4).
$$

The relation (1) is expressed in the fact that this map lands in the fiber of (2). For any n, and any CW complex X, the map

$$
[X, \mathrm{B}\,\mathrm{PU}_n[4]] \to \mathrm{H}^2(X, \mathbb{Z}/n) \times \mathrm{H}^4(X, \mathbb{Z})
$$

is an injection with image precisely the classes (x, y) satisfying (1).

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The 5th stage of the Postnikov tower for $B\,PU_2$ gives an extension,

$$K(\mathbb{Z}/2, 5) \to B\,PU_2[5] \to B\,PU_2[4],$$

which is classified by a class $u \in H^6(B\,PU_2[4], \mathbb{Z}/2)$. Given a space X and classes $(x, y) \in H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})$ satisfying (1), one has a uniquely determined map $f : X \to B\,PU_2[4]$, and hence a cohomology class $f^*(u)$. In order for f to lift to a map $X \to B\,PU_2[5]$, it is necessary for $f^*(u) = 0$. If $\dim X \leq 6$, this is also a sufficient condition. If f is determined by classes (x, y), write $u(x, y)$ for $f^*(u)$ in $H^6(X, \mathbb{Z}/2)$.

Theorem 1. *Let X be a 6-dimensional CW complex. The image of $[X, B\,PU_2] \to H^2(X, \mathbb{Z}/2) \times H^4(X, \mathbb{Z})$ is the set of classes (x, y) such that $u(x, y) = 0$. Moreover, there is a 6-dimensional CW complex X and classes (x, y) such that $u(x, y) \neq 0$.*

Proof. The discussion above proves the first statement. To prove the second statement is equivalent to showing that the extension $K(\mathbb{Z}/2, 5) \to B\,PU_2[5] \to B\,PU_2[4]$ is non-split. Indeed, if it is non-split, then the 6-skeleton of $B\,PU_2[4]$ together with the composition

$$\mathrm{sk}_6\,(B\,PU_2[4]) \to B\,PU_2[4] \to K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}, 4)$$

gives an example.

The quotient map $SU_2 \to PU_2$ induces a map on classifying spaces $BSU_2 \to B\,PU_2$, which induces an isomorphism on homotopy groups π_i for $i > 2$. By the naturality of Postnikov towers, there is thus a map of extensions

$$
\begin{array}{ccccc}
K(\mathbb{Z}/2, 5) & \longrightarrow & BSU_2[5] & \longrightarrow & K(\mathbb{Z}, 4) \\
\| & & \downarrow & & \downarrow \\
K(\mathbb{Z}/2, 5) & \longrightarrow & B\,PU_2[5] & \longrightarrow & B\,PU_2[4].
\end{array}
$$

If the class of the extension in $H^6(K(\mathbb{Z}, 4), \mathbb{Z}/2)$ is non-zero, then by the commutativity of the diagram, the class in $H^6(B\,PU_2[4], \mathbb{Z})$ is non-zero. It is not hard to show, using the Serre spectral sequence, that $H^6(K(\mathbb{Z}, 4), \mathbb{Z}/2) = \mathbb{Z}/2$, generated by a class γ. On the other hand, $H^*(BSU_2, \mathbb{Z}) = \mathbb{Z}[c_2]$, where the class c_2 has degree 4. Therefore, $H^6(BSU_2, \mathbb{Z}/2) = 0$. Since $BSU_2 \to BSU_2[5]$ is a 6-equivalence, it follows that $H^6(BSU_2[5], \mathbb{Z}/2) = 0$ as well. If the extension were split, then the pullback of γ to $BSU_2[5]$ would be non-zero. Thus the extension is not split. $\qquad\square$

In [2], we show that for a 6-dimensional CW complex X and a fixed non-zero class $x \in H^2(X, \mathbb{Z}/2)$, it is possible for the set of $y \in H^4(X, \mathbb{Z})$ such that (x, y) satisfies (1) to be non-empty, while the set of maps $\xi : X \to B\,PU_2$ such that $t(\xi) = x$ is empty. Moreover, we can take X to be the complex points of a smooth affine 6-fold over $\mathbb{C}$. Thus, in some sense, Woodward's statement can fail as badly as possible in some situations.

Now, we prove a corollary, which amounts to determining the class u in $H^6(B\,PU_2[4], \mathbb{Z}/2)$. By Serre [4, Section 9], the $\mathbb{Z}/2$-cohomology of $K(\mathbb{Z}/2, 2)$ is a polynomial ring

$$H^*(K(\mathbb{Z}/2, 2), \mathbb{Z}/2) = \mathbb{Z}/2[u_2, \mathrm{Sq}^1 u_2, \mathrm{Sq}^2 \mathrm{Sq}^1 u_2, \ldots, \mathrm{Sq}^{2^k} \mathrm{Sq}^{2^{k-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1 u_2, \ldots],$$

where u_2 is the fundamental class in degree 2, and Sq^i denotes the ith Steenrod operation. Let $B\,PU_2[4] \to K(\mathbb{Z}/2, 2)$ be denoted by p.

Corollary 2. *The set $\{u, p^*u_2^3, p^*(\mathrm{Sq}^1 u_2)^2\}$ forms a basis of the 3-dimensional $\mathbb{Z}/2$-vector space $H^6(B\,PU_2[4], \mathbb{Z}/2)$, and this characterizes u up to homotopy.*

Proof. First, recall the exceptional isomorphism $PU_2 \cong SO_3$. Thus, $H^*(B\,PU_2, \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]$, where w_i has degree i. The map $B\,PU_2 \to B\,PU_2[2] \simeq K(\mathbb{Z}/2, 2)$ is a 4-equivalence, so that w_2 is the pullback of u_2, and w_3 is the pullback of $\mathrm{Sq}^1 u_2$. It follows, in fact, that $H^*(B\,PU_2[k], \mathbb{Z}/2)$ contains the algebra $\mathbb{Z}/2[w_2, w_3]$ for $n \geq 2$. A brief examination of the Serre spectral sequence for the fibration $K(\mathbb{Z}, 4) \to B\,PU_2[4] \to K(\mathbb{Z}/2, 2)$ shows that the dimension of $H^6(B\,PU_2[4], \mathbb{Z}/2)$ is at most 3. The classes w_2^3 and w_3^2 must

survive and be distinct, since they do in the cohomology of $\mathrm{B\,PU_2}$. Finally, since we showed in the proof of the theorem that the extension class u restricts to the non-zero class in $\mathrm{H}^6(K(\mathbb{Z}, 4), \mathbb{Z}/2)$, it follows that the asserted classes form a basis for $\mathrm{H}^6(\mathrm{B\,PU_2}[4], \mathbb{Z}/2)$, as desired. $\qquad\square$

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